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by

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The Florida State University

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ABSTRACT

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An Inequality for Convex Functions Involving G-Majorization

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In this note we derive a simple inequality involving expectations of convex functions and the notion of G-majorization. The result extends a similar inequality of Marshall and Proschan (1965) involving majorization. A number of useful applications of the more general inequality are then presented.

Let G be a group of matrices (linear transformations) acting on R^n. A vector $\underset{\sim}{a} = (a_1,\ldots,a_n)$ is said to G-majorize a vector $\underset{\sim}{b} = (b_1,\ldots,b_n)$, written $\underset{\sim}{a} \overset{G}{\geq} \underset{\sim}{b}$, if $\underset{\sim}{b}$ is in the convex hull of the G-orbit of $\underset{\sim}{a}$. If $G = P_n$, the group of permutation matrices, G-majorization coincides with majorization (see Eaton and Perlman, 1976). A random vector $\underset{\sim}{X} = (X_1,\ldots,X_n)$ is said to be G-invariant if $\underset{\sim}{X}$ is stochastically equal to $g\underset{\sim}{X}$ for all $g\varepsilon G$. When $G = P_n$, we say that $X_1,\ldots,X_n$ are exchangeable random variables. For vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, let $\underset{\sim}{a}\cdot\underset{\sim}{b} \overset{\text{def}}{=} (a_1b_1,\ldots,a_nb_n)$.

Theorem 1. Let G be a finite group such that for all $g\varepsilon G$ there exist h and $k\varepsilon G$ for which $h(g\underset{\sim}{a}\cdot\underset{\sim}{b}) = \underset{\sim}{a}\cdot k\underset{\sim}{b}$ for all vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Let $\underset{\sim}{X}$ be a G-invariant random vector, ϕ a continuous, convex, G-invariant function and $\underset{\sim}{a} \overset{G}{\geq} \underset{\sim}{b}$. Then

(1) $E\phi(\underset{\sim}{a}\cdot\underset{\sim}{X}) \geq E\phi(\underset{\sim}{b}\cdot\underset{\sim}{X})$.

Moreover, if ϕ is strictly convex, equality holds only when $\underset{\sim}{a} = g\underset{\sim}{b}$ for some $g\varepsilon G$, or when the X_i are all zero with probability one.

Proof. Let $G = \{g_i\}_{j=1}^{m}$. Then we may write $\underset{\sim}{b} = \sum_{j=1}^{m} \alpha_j g_j \underset{\sim}{a}$, where each $\alpha_j \geq 0$ and $\sum_{j=1}^{m} \alpha_j = 1$. It follows that $E\phi(\underset{\sim}{b}\cdot\underset{\sim}{X}) = E\phi([\sum_{j=1}^{m} \alpha_j g_j \underset{\sim}{a}]\cdot\underset{\sim}{X}) =$

$E\phi(\sum_{j=1}^{m} \alpha_j[g_j\underset{\sim}{a}\cdot\underset{\sim}{X}]) \le \sum_{j=1}^{m} \alpha_j E\phi(g_j\underset{\sim}{a}\cdot\underset{\sim}{X})$. For each j, let h_j and k_j be the elements of G for which $h_j(g_j\underset{\sim}{a}\cdot\underset{\sim}{X}) = \underset{\sim}{a}\cdot k_j\underset{\sim}{X}$. Then $E\phi(g_j\underset{\sim}{a}\cdot\underset{\sim}{X}) = E\phi(h_j(g_j\underset{\sim}{a}\cdot\underset{\sim}{X}))$ [by the G-invariance of ϕ] $= E\phi(\underset{\sim}{a}\cdot k_j\underset{\sim}{X}) = E\phi(\underset{\sim}{a}\cdot\underset{\sim}{X})$ [by the G-invariance of $\underset{\sim}{X}$]. Thus $E\phi(\underset{\sim}{b}\cdot\underset{\sim}{X}) \le \sum_{j=1}^{m} \alpha_j E\phi(\underset{\sim}{a}\cdot\underset{\sim}{X}) = E\phi(\underset{\sim}{a}\cdot\underset{\sim}{X})$.

In case ϕ is strictly convex, it is clear from the above proof that equality holds only if for some $g\epsilon G$, $\underset{\sim}{b}\cdot\underset{\sim}{X} = g\underset{\sim}{a}\cdot\underset{\sim}{X}$ with probability one. □

Remark 1. If $G = P_n$, then for all $g\epsilon G$ and vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $g^{-1}(g\underset{\sim}{a}\cdot\underset{\sim}{b}) = \underset{\sim}{a}\cdot g^{-1}\underset{\sim}{b}$. Therefore in this special case the hypothesis of Theorem 1 is satisfied. It follows that the main result of Marshall and Proschan (1965) involving majorization is a special case of Theorem 1.

Remark 2. Other groups of interest for which the hypothesis of the theorem is satisfied are: (a) The group G_1 of sign changes and (b) the group G_2 of permutations and sign changes, as is readily verified.

Remark 3. Note that in $G_2, g^{-1}(g\underset{\sim}{a}\cdot\underset{\sim}{b}) \ne \underset{\sim}{a}\cdot g^{-1}\underset{\sim}{b}$ for all $g\epsilon G_2$. For example if $g = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ then $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\underset{\sim}{a}\cdot\underset{\sim}{b}\right) = \underset{\sim}{a}\cdot\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\underset{\sim}{b}\right)$. So in G_2 the milder requirement is needed that for all $g\epsilon G$ there exist h and $k\epsilon G$ for which $h(g\underset{\sim}{a}\cdot\underset{\sim}{b}) = \underset{\sim}{a}\cdot k\underset{\sim}{b}$. Also note that this condition is not satisfied for some groups. For example, if $G = \left\{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, I\right\}$ and $g = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, then clearly there do not exist h and k for which $h(g\underset{\sim}{a}\cdot\underset{\sim}{b}) = \underset{\sim}{a}\cdot k\underset{\sim}{b}$.

Remark 4. Let $G = G_2$ and let vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have all components non-negative. Then $\underset{\sim}{a} \overset{G}{\le} \underset{\sim}{b}$ if and only if $\underset{\sim}{a}$ is weakly majorized by $\underset{\sim}{b}$. (See Marshall, Walkup and Wets (1967) for the definition of weak majorization.)

Similarly, let $G = G_1$, and let vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have all components nonnegative. Then $\underset{\sim}{a} \overset{G}{\leq} \underset{\sim}{b}$ if and only if $a_i \leq b_i$ for $i = 1,2,\ldots,n$. It follows that Theorem 1 yields results concerning weak majorization and the usual partial ordering of the plane. (See also Remark 8.)

Remark 5. For comments on a converse to Theorem 1, see Remark 3 of Marshall and Proschan (1965). Also see Remark 4 of that paper for a counterexample showing that the conclusion of Theorem 1 need not necessarily hold when we weaken the hypothesis to require ϕ to be only continuous and isotone with respect to the G-majorization ordering, i.e., G-monotone. (A G-invariant convex function is necessarily G-monotone). However, by using a path lemma of Eaton and Perlman (1976), it is possible to show that if G is a reflection group, then Theorem 1 holds when ϕ is merely continuous and convex along all the line segments joining $\underset{\sim}{a}$ with $g\underset{\sim}{a}$ for all $g \varepsilon G$. (See Eaton and Perlman (1976) for the definition of a reflection group.) This is consistent with Remark 4 of Marshall and Proschan (1965). Thus if $G = G_2$ we need only require that ϕ, considered as a function of a specified pair of coordinates with all other coordinates held fixed, be convex. Note that this condition on ϕ is the same as that in Remark 4 of Marshall and Proschan (1965). Similarly, if $G = G_1$, we need only require that ϕ, considered as a function of a specified coordinate with all other coordinates held fixed, be convex.

Corollary 1. Let $G = G_2$. Let $X(\sigma_1),\ldots,X(\sigma_n)$ be independent random variables, where $X(\sigma_i)$ is normally distributed with mean zero and standard deviation σ_i. Let ϕ be continuous, convex, and G-invariant, and $(\sigma_1,\ldots,\sigma_n) \overset{G}{\geq} (\sigma_1',\ldots,\sigma_n')$. Then

$$(2) \quad E\phi(X(\sigma_1),\ldots,X(\sigma_n)) \geq E\phi(X(\sigma_1'),\ldots,X(\sigma_n')).$$

Proof. Let $Y_1,\ldots,Y_n$ be independently distributed standard normal random variables. Then $E\phi(\sigma_1 Y_1,\ldots,\sigma_n Y_n) \ge E\phi(\sigma_1' Y_1,\ldots,\sigma_n' Y_n)$ by Theorem 1. Since $\sigma_i Y_i$ and $X(\sigma_i)$ have the same distribution, the result follows. □

Remark 6. Similar results are true when G is P_n or G_1.

Remark 7. Note that the only property of $X(\sigma_i)$ used in the proof of Corollary 1 is that $X(\sigma_i)$ and $\sigma_i Y$ have the same distribution where Y is a random variable distributed symmetrically about zero. Thus, for example, Corollary 1 is still true when $X(\sigma_i)$ is uniformly distributed on the interval $(-\sigma_i,\sigma_i)$. For simplicity, Corollary 1 is stated for the special case $X(\sigma_i)$ is normal.

Remark 8. Note that since $\underset{\sim}{a} \overset{P_n}{\le} \underset{\sim}{b}$ or $\underset{\sim}{a} \overset{G_1}{\le} \underset{\sim}{b}$ implies $\underset{\sim}{a} \overset{G_2}{\le} \underset{\sim}{b}$, (2) holds when $\underset{\sim}{a} \overset{P_n}{\le} \underset{\sim}{b}$ or $\underset{\sim}{a} \overset{G_1}{\le} \underset{\sim}{b}$. (A similar remark applies whenever the G_2 ordering holds.)

Corollary 2. Let ϕ be continuous, convex, and invariant under permutations and sign changes, and let $\underset{\sim}{a} \overset{G_2}{\ge} \underset{\sim}{b}$. Then

(3) $\sum^* \phi(a_1x_1,\ldots,a_nx_n) \ge \sum^* \phi(b_1x_1,\ldots,b_nx_n)$ where $\sum^*$ denotes summation over all sign changes and permutations of the x_i.

Proof. (3) is an immediate consequence of (1), where $P\{(x_1,\ldots,x_n) = ((-1)^{\sigma_1}x_{i_1},\ldots,(-1)^{\sigma_n}x_{i_n})\} = \frac{1}{2^n n!}$, where $(i_1,\ldots,i_n)$ is a permutation of $(1,2,\ldots,n)$ and $\sigma_i = 0$ or 1 for $i=1,\ldots,n$.

Remark 9. Corollary 2 is a variation of Muirhead's Theorem. (See Hardy, Littlewood, and Pólya, 1952, pp. 44-48.)

Remark 10. For other possible applications yielding inequalities, see Marshall and Proschan (1965).

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